ISRAEL JOURNAL OF MATHEMATICS, Vol. 55, No. 3, 1986

ON P.I. RING AND STANDARD IDENTITIES

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ABSTRACT

An example is given of a ring R (with 1) satisfying the standard identity $S_6[x_1,\dots,x_6]$ but $M_2(R)$, the 2×2 matrix ring over R, does not satisfy $S_{12}[x_1,\dots,x_{12}]$. This is in contrast to the case $R = M_n(F)$, F a field, where by the Amitsur–Levitzki theorem R satisfies $S_{2n}[x_1,\dots,x_{2n}]$ and $M_2(R)$ satisfies $S_{4n}[x_1,\dots,x_n]$.

§0. Introduction

The fundamental theorem of Amitsur–Levitzki states that $R = M_n(F)$, the $n\times n$ matrix ring over a commutative ring F, satisfies the standard polynomial identity $S_{2n}[x_1,\dots,x_{2n}]$ (e.g., [7, p. 21]). Consequently $M_{2n}(F) = M_2(R)$ satisfies the standard identity $S_{4n}[x_1,\dots,x_{4n}]$. This leads to the following natural question:

Let R be a ring ($1\in R$) satisfying $S_{2n}[x_1,\dots,x_{2n}]$, does $M_2(R)\equiv\left(\begin{smallmatrix}R & R\\ R & R\end{smallmatrix}\right)$ satisfy $S_{4n}[x_1,\dots,x_{4n}]$?

Regev [6, p. 506] obtained a positive result to an analogous question; he shows

THEOREM. *Let R satisfy c_{k+1}, the $k+1$ Capelli polynomial. Then $M_2(R)$ satisfies c_{4k+1}.*

In fact he proves much more: If S is any Q-algebra satisfying c_{l+1}, then $R\otimes S$ satisfies c_{kl+1}.

This is an analogous result since the minimal Capelli polynomial which $R = M_n(F)$ satisfies is C_{n^2+1}; now replacing n by $2n$ will give $c_{(2n)^2+1} = c_{4n^2+1}$, the minimal Capelli which $M_2(R)$ satisfies.

[†] Part of this work was done while the author enjoyed the hospitality of the University of California at San Diego, the University of Texas at Austin and the University of Washington at Seattle.

Received June 7, 1985 and in revised form April 20, 1986

We shall show that the answer to the above-mentioned question is negative. In fact we have

THEOREM. *There exists a finite dimensional algebra R over a field F* (char $F \neq 2,5$), *satisfying $S_6[x_1,\dots,x_6]$, but $M_2(R)$ does not satisfy the standard identity $S_{12}[x_1,\dots,x_{12}]$.*

The proof depends upon the identity $S_6^*[x_1^{(1)},x_2^{(2)},x_3,\dots,x_6]$ (see §1 for definition), its vanishing on $M_2(F)$ and on certain results in [1, Thm. 1.15], where $S_m^*[x_1^{(1)},x_2^{(2)},x_3,\dots,x_6]$ was originally defined.

As a corollary we obtain a counter example to the "generalized Bergman and Small question" ([1, 1.17]). More precisely, we have

THEOREM. *There exists a finite dimensional algebra R, over a field F, $1 \in R$, such that*

(i) *R satisfies $S_6[x_1,\dots,x_6]$,*

(ii) p.i.d. $(R/M)=2$ *for every maximal M in R,*

but R is not an Azumaya algebra.

In fact the algebra R appearing in the previous theorem will do.

§1. Proof of main result

Let $S_m[x_1,\dots,x_m]=\Sigma_{\sigma\in\Sigma_m}(-1)^\sigma x_{\sigma(1)}\cdots x_{\sigma(m)}$ be the m-th standard identity, where Σ_m denotes the symmetric group on m letters and $(-1)^\sigma$ denotes sg σ, the signature of $\sigma\in\Sigma_m$.

Let $S_m^*[x_1^{(1)},x_2^{(2)},\dots,x_m]$ denote the sum of all the terms in $S_m[x_1,\dots,x_m]$ where x_1 appears to the left of x_2. We have

$$(1)\qquad S_m^*[x_1^{(1)},x_2^{(2)},\dots,x_m]-S_m^*[x_2^{(1)},x_1^{(2)},x_3,\dots,x_m]=S_m[x_1,\dots,x_m],$$

$$(2)\qquad \begin{aligned}&S_m^*[x_1^{(1)},x_2^{(2)},x_3,\dots,x_m]\\ &=\sum_{i\neq 1,2}(-1)^{i-1}x_iS_{m-1}^*[x_1^{(1)},\dots,x_1,\dots,x_m]+x_1S_{m-1}[x_2,\dots,x_m],\end{aligned}$$

$$\begin{aligned}&S_m^*[x_1^{(1)},x_2^{(2)},x_i,\dots,x_m]\\ &=\sum_{i\neq 1,2}(-1)^{m-1}S_{m-1}^*[x_1^{(1)},\dots,x_i,\dots,x_m]x_i(-1)^mS_{m-1}[x_1,x_3,\dots,x_m]x_2;\end{aligned}$$

this is just the analog of the Laplace expansion of $S_m[x_1,\dots,x_m]$.

$$(3)\qquad S_m^*[x_1^{(1)},x_2^{(2)},x_{\sigma(3)},\dots,x_{\sigma(m)}]=(-1)^\sigma S_m^*[x_1^{(1)},x_2^{(2)},x_3,\dots,x_m]$$

where $\sigma \in \Sigma_{m-2}$. This is proved exactly as the analogous statement for $S_m[x_1, \ldots, x_m]$ (e.g. [5], p. 16). $S^*_m[x_1^{(1)}, x_2^{(2)}, x_3, \ldots, x_m]$ was first introduced in [1] where the following is proved (Theorem 1.15(i)).

PROPOSITION 1. *Let R be an F-algebra, $1 \in R$, and satisfying S_{2n}. Suppose that for every maximal ideal M in R, R/M does not satisfy $S_{2(n-3)}$, where $n \geqq 6$ and* char $F \neq 2, 5$. *Then R is Azumaya. If* $\operatorname{rank}(R/M) = k^2$ *for every maximal ideal M in R, then* $\operatorname{rank}(R) = k^2$.

The next lemma is a technical result we use, the proof of which we postpone to the end.

LEMMA 2. *Let F be a commutative ring and $M_2(F)$ be the 2×2 matrix ring over F. Then $S^*_6[x_1, \ldots, x_6]$ vanishes on $M_2(F)$.*

We shall use $N(R)$ for the nil radical of R and $M_d(R)$ for the $d \times d$ matrices over R.

PROPOSITION 3. *Let R be a finite dimensional F algebra, $1 \in R$, such that $R/N(R) \cong M_d(F)$ and satisfying $N(R)^3 = \{0\}$. Then R satisfies $S_m[x_1, \ldots, x_m]$ provided $R/N(R)$ satisfies $S^*_m[x_1^{(1)}, x_2^{(2)}, x_3, \ldots, x_m]$.*

PROOF. By the Wederburn principal theorem $R \cong A \oplus N(R)$, where $A \cong M_d(F)$. Also by [2, p. 54] $N(R) = N^A \otimes_F A$, where $N^A = \{n \in N(R) \mid na = an, a \in A\}$. Consequently $R \cong M_d(F \oplus N^A) = M_d(F) \otimes_F (F \oplus N^A)$.

We shall prove that $S_m[x_1, \ldots, x_m] = 0$. By the linearity of $S_m[x_1, \ldots, x_m]$ we may assume that $x_i = a_i c_i$ where $a_i \in A$, and $c_i \in F$ or $c_i \in N^A$, $i = 1, \ldots, m$. If $c_{i_1}, c_{i_2}, c_{i_3} \in N^A$ for some three indices, then $N^3 = \{0\}$ implies the vanishing of $S_m[x_1, \ldots, x_m] = 0$. Suppose that two of the indices c_{i_1}, c_{i_2} are in N^A, say $i_1 = 1$, $i_2 = 2$, Thus, by (1)

$$\begin{aligned} S_m[x_1, \ldots, x_m] &= S^*_m[x_1^{(1)}, x_2^{(2)}, x_3, \ldots, x_m] - S^*_m[x_2^{(1)}, x_1^{(2)}, x_3, \ldots, x_m] \\ &= c_1 c_2 S^*_m[a_1^{(1)}, a_2^{(2)}, c_3 a_3, \ldots, c_m a_m] - c_2 c_1 S^*[a_2^{(1)}, a_1^{(2)}, c_3 a_3, \ldots, c_m a_m] \\ &= 0 \end{aligned}$$

where the last equality holds since $S^*_m[x_1^{(1)}, x_2^{(2)}, x_3, \ldots, x_m]$ vanishes on A, and observing that $x_i = a_i c_i \in A$ for $i = 3, \ldots, m$. If only one of the c_i's is in N^A, e.g. $i = 1$, then, $x_i \in A$ for $i = 2, \ldots, m$ and consequently

$$S_m[x_1, \ldots, x_m] = S_m[c_1 a_1, c_2 a_2, \ldots, c_m a_m] = c_1 S_m[a_1, c_2 a_2, \ldots, c_m a_m] = 0,$$

where the last equality holds since $S^*_m[x_1^{(1)}, x_2^{(2)}, x_3, \ldots, x_m]$ vanishes on A and by

(1) so does $S_m[x_1, \ldots, x_m]$. Finally if $c_1, \ldots, c_m$ are all in A, then $x_1, \ldots, x_m$ are in A and the previous reasoning holds.

We shall now prove, using Theorem 2, our main result.

THEOREM 4. *There exists a ring R, $1 \in R$, satisfying the standard identity $S_6[x_1, \ldots, x_6]$ but $M_2(R)$, the 2×2 matrix over R, does not satisfy $S_{12}[x_1, \ldots, x_{12}]$.*

PROOF. Let C be a finite dimensional algebra over a field F (char $F \neq 2,5$), satisfying the following properties.

(i) $1 \in C$, C is *not* commutative.

(ii) $N(C)^3 = \{0\}$, $N(C)^2 \neq \{0\}$, and $N(C)$ is the unique maximal ideal of C.

(iii) $C/N(C) \cong F$.

It is an easy exercise to produce such a C. Let $R = M_2(C)$. We have $R = M_2(F) \otimes_F C$. Observe that

$$N(R) = M_2(N(C)) = \begin{pmatrix} N(C) & N(C) \\ N(C) & N(C) \end{pmatrix}.$$

Consequently $N(R)^3 = \{0\}$. Also $R/N(R) \cong M_2(F)$ and therefore satisfying $S_6^*[x_1^{(1)}, x_2^{(2)}, x_3, \ldots, x_6]$. Now Proposition 3 implies that R satisfies $S_6[x_1, \ldots, x_6]$.

We shall now show that $M_2(R)$ does not satisfy $S_{12}[x_1, \ldots, x_{12}]$. Indeed $N(R)$ being the unique maximal ideal of R, implies that $M_2(R)$ has a *unique* maximal ideal

$$M_2(N(R)) = \begin{pmatrix} N(R) & N(R) \\ N(R) & N(R) \end{pmatrix} = N(M_2(R)),$$

and $M_2(R)/M_2(N(R)) \cong M_4(F)$.

Suppose that $M_2(R)$ satisfies $S_{12}[x_1, \ldots, x_{12}]$. Then by Proposition 1 ($n = 6$) and the previous isomorphism, $M_2(R)$ is an Azumaya algebra of constant rank 4^2. Consequently, R is an Azumaya algebra of constant rank 4. This implies by [3, p. 160] that C is commutative, a contradiction.

PROOF OF LEMMA 2. Given $x_1, \ldots, x_6$ in $M_2(F)$ we may assume, by the linearity of $S_6^*[x_1^{(1)}, x_2^{(2)}, x_3, x_4, x_5, x_6]$, that x_i is a basis element for $i = 1, \ldots, 6$. Also by (3) if $x_i = x_j$ for $i \neq j$ and $i, j \geqq 3$ then $S_6^*[x_1^{(1)}, x_2^{(2)}, x_3, \ldots, x_6] = 0$. Moreover (3) implies that if $S_6^*[x_1^{(1)}, x_2^{(2)}, x_3, x_4, x_5, x_6] = 0$ for some choice of x_3, x_4, x_5, x_6 then $S_6^*[x_1^{(1)}, x_2^{(2)}, x_{\sigma(3)}, x_{\sigma(4)}, x_{\sigma(5)}, x_{\sigma(6)}] = 0$, for every permutation σ. Consequently we need only check that $S_6^*[x_1^{(1)}, x_2^{(2)}, e_{11}, e_{12}, e_{21}, 1] = 0$ ($e_{11}, e_{12}, e_{21}, 1$ is a basis for $M_2(F)$).

Finally, as for the usual standard identity (e.g., [7, p. 103, Ex. 1.2(3)]), one has that for every even m, $S^*_m[x_1^{(1)}, x_2^{(2)}, x_3, \ldots, x_{m-1}, 1] \equiv 0$, in particular $S^*_6[x_1^{(1)}, x_2^{(2)}, e_{11}, e_{12}, e_{21}, 1] = 0$.

Acknowledgement

I would like to thank T. Stafford and the referee for supplying us short proofs of Lemma 2.

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